

OPTION PRICING,
BAYES RISK
AND
APPLICATIONS

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Summary

A statistical decision problem is hidden in the core of option pricing. A simple form for the price C of a European call option is obtained via the minimum Bayes risk, R_B , of a 2-parameter estimation problem, thus justifying calling C Bayes (B-)price. The result provides new insight in option pricing, among others obtaining C for some stock-price models using the underlying probability instead of the risk neutral probability and giving R_B an economic interpretation. When logarithmic stock prices follow Brownian motion, discrete normal mixture and hyperbolic Lévy motion the obtained B-prices are “fair” prices. A new expression for the price of American call option is also obtained and statistical modeling of R_B can be used when pricing European and American call options.

Keywords: American option, Bayes risk, European option, Hyperbolic Lévy motion, Leverage, Normal mixture models, Risk neutral probability

1 Introduction

So far, the core of the option pricing problem is a topic studied mainly in Finance and Mathematical Finance. This work shows, surprisingly, that option pricing can be seen as a statistical decision problem with some useful implications.

By purchasing at time t_0 for premium C a share's European call option, the buyer has in the future, at time $T(> t_0)$, the option to buy the share at predetermined fixed value X . The “fair” price C is obtained, under some assumptions, by “replicating the call,” i.e. by creating a portfolio that matches the call's payoff at T (Black and Scholes, 1973, Merton, 1973). This procedure guarantees C does not allow arbitrage, i.e. that the call option's buyer cannot make profit with probability 1. However, “the procedure may be tedious and computationally demanding” (Sundaram, 1997, p. 85) . Alternatively, C is obtained by discounting at t_0 the expected value of the call's payoff at T under the risk neutral (or equivalent martingale) probability that is not always easy to determine. The interested reader may refer to Sundaram (1997) for a rather informal and accessible introduction to the use and determination of risk neutral probability.

More complex financial instruments have been introduced and priced in the sequel, as for example the American call option, where the option's holder

may exercise the right to buy the share at price X any time $t \in (t_0, T]$; see Hull (1993) for other types of options. So far, for the European, the American and other call options, no statistical problems have been determined whose solutions provide the corresponding “fair” price.

Using the equivalent martingale probability approach, a simple, new expression for the price C of a European call option is obtained herein, which involves the minimum Bayes risk, R_B , of a 2-parameter statistical estimation problem and some known quantities. In this way option pricing can be seen as the solution of a statistical problem. In fact, C increases when the corresponding statistical estimation problem becomes simpler. The result suggests discounting stock prices with expectations’ ratios. C is calculated via R_B for various stock price models, circumventing in some cases the search for the equivalent martingale probability and complementing the approach by Gerber and Shiu (1994) when the martingale probability is not unique. For the trader selling the call option, R_B is a lower bound on the ratio of its liability and its expected assets at t_0 , called “accounting leverage”. R_B can be used when calculating the price of an American option.

The Bayesian approach in this work and the results are not related with Bayesian calibration used to mark the stock price model to market obtaining posterior distributions of model parameters for a given prior (see, for example,

Gupta and Reisinger, 2011). For a different statistical approach to theoretical and practical issues in stock trading, including option pricing, the interested reader may consult Franke, J. et al. (2010).

In section 2, a quantitative description of the results is presented. In section 3, the new expression for C is obtained via R_B . In the applications in sections 4 and 5, C is obtained via R_B when the logarithmic stock prices follow Brownian motion, discrete normal mixture and hyperbolic Lévy motion. Proofs are in the Appendix and Figure 1 appears after the references.

2 A quantitative description of the results

Let S_t be the price of the stock at time t , $0 < t_0 \leq t \leq T$, $S_{t_0} = s_{t_0}$, let (Ω, \mathcal{F}, P) be the underlying probability space and let P^* be the (assumed for now) unique martingale probability equivalent to P . The buyer of a European call option at t_0 has the right to buy one share at time T with “strike” price X by paying “fair” price C , i.e. the discounted, P^* -expected cost at maturity. An expression for C is obtained that involves cumulative distributions F_1^* and F_0^* defined via P^* ; F_0^* is the cumulative distribution of $\ln \frac{S_T}{ES_T}$ under P^* and F_1^* is an equivalent probability determined in (5). Distributions F_1^* and F_0^* constitute the parameter space of the statistical decision problem that

determines the option's price. It is shown that

$$C = s_{t_0} - R_B(s_{t_0} + Xe^{-r(T-t_0)}), \quad (1)$$

with R_B the minimum Bayes risk for the estimation of F_1^* and F_0^* with 0 – 1 loss, $0 < R_B < 1$, $r = \ln(1 + i)$, i fixed interest. From (1) it follows that when the difficulty of the estimation problem decreases by increasing the Hellinger distance between f_0^* and f_1^* , R_B decreases and the C -value increases. For example, under the Black-Scholes-Merton (B - S - M) assumptions for the stock price process, when the volatility increases the Hellinger distance between f_0^* and f_1^* increases, therefore the difficulty of the estimation problem decreases and hence the C -value increases. It also follows that C can be obtained from a simple game with loss, profit and respective probabilities determined by R_B .

At t_0 , accounting leverage for the trader writing the call option by taking a loan can be measured with the ratio

$$\frac{s_{t_0} - C}{s_{t_0} + Xe^{-r(T-t_0)}P(S_T > X)} \geq R_B = \frac{s_{t_0} - C}{s_{t_0} + Xe^{-r(T-t_0)}}. \quad (2)$$

The numerator in the left side of (2) is the trader's liability and the denominator its total expected assets, both at t_0 . Since $P(S_T > X)$ is unknown, the trader or the bank providing the amount $s_{t_0} - C$ may not allow the transaction when R_B is “high” .

The results justify naming C Bayes (B -) price denoted by $\tilde{C}_{B,t_0}(P^*) = C$

and indicate a new expression for the price of an American option in Karatzas (1988, p. 50, equation 5.10 with parameter values those in Example 4.4, p. 44). Advantages of (1) include its simplicity and the possibility of modeling R_B in order to obtain C or its approximation for various choices of F_0^* and F_1^* .

Motivated by these findings and the definition of F_0^* , we discount stock prices with expectation ratios and it is shown under B - S - M assumptions that B -price $\tilde{C}_{B,t_0}(P)$ is B - S - M price C obtained under P^* . This is not surprising since the expectations ratio discounted prices $\{\frac{E_P S_{t_0}}{E_P S_t} S_t = \frac{s_{t_0}}{E_P S_t} S_t, t \geq t_0\}$ are martingale under P . Note also that for the equivalent martingale probability P^* of any P , $\frac{s_{t_0}}{E_{P^*} S_t} = e^{-r(t-t_0)}$ i.e. the usual discounting factor.

Since discounted price $s_{t_0} S_T / E_P S_T$ is used when calculating B -price, sufficient conditions are provided for the mean-adjusted process $\{S_t / E_P S_t, t > 0\}$ of geometric prices to be martingale under P . These conditions hold for all t when $\ln(S_t / E_P S_t)$ is Brownian or hyperbolic Lévy motion, but for discrete normal mixture $\{S_t / E_P S_t, t > 0\}$ is “nearly” martingale for small t , which is sufficient to obtain the option’s price. B -price for the latter, with small T -values, indicates overpricing when using instead B - S - M price for normal distribution with the same mean and variance. B -price for Lévy motion is a “fair” price complementing the prices obtained by Eberlein and Keller (1995)

using the Esscher transform (Gerber and Shiu, 1994).

3 The price of a European call and Bayes risk

SET-UP (A): S_t is the stock's price at time t on the probability space (Ω, \mathcal{F}, P) , $0 < t_0 \leq t \leq T, S_{t_0} = s_{t_0}$; X is the strike price at maturity T ; P^ is the unique equivalent martingale probability to P ; $r = \ln(1 + i)$, i is fixed interest; probabilities π_1, π_0 are $\pi_1 = \frac{s_{t_0}}{s_{t_0} + X e^{-r(T-t_0)}} = 1 - \pi_0$; expectation EU is obtained under P^* . The "fair" price C of a European call option is the discounted expected value of the call's payoff at maturity under P^* :*

$$C = e^{-r(T-t_0)} E(S_T - X)I(S_T > X); \quad (3)$$

I denotes indicator function.

Denote by F_0^* the c.d.f. of $Y = \ln \frac{S_T}{ES_T}$ under P^* ,

$$P^*(Y \leq y) = P^*\left[\ln \frac{S_T}{ES_T} \leq y\right] = F_0^*(y), \quad F_0^{*'}(y) = f_0^*(y), \quad -\infty < y < +\infty. \quad (4)$$

Observe that

$$1 = E \frac{S_T}{ES_T} = E e^{\ln \frac{S_T}{ES_T}} = E e^Y = \int_{-\infty}^{+\infty} e^y f_0^*(y) dy,$$

thus for

$$f_1^*(y) = e^y f_0^*(y), \quad F_1^{*'}(y) = f_1^*(y), \quad -\infty < y < +\infty, \quad (5)$$

it follows that f_1^* is density with cumulative distribution function F_1^* and the mean value under F_0^* is smaller than that under F_1^* .

Proposition 3.1 *Under (A), (4) and (5), for the “fair” price C of the European call option at t_0 it holds*

a)

$$\frac{s_{t_0} - C}{s_{t_0} + Xe^{-r(T-t_0)}} = \pi_1 F_1^*(-D) + \pi_0 [1 - F_0^*(-D)] = \inf_{d>0} [\pi_1 F_1^*(\ln \frac{d}{ES_T}) + \pi_0 [1 - F_0^*(\ln \frac{d}{ES_T})]], \quad (6)$$

with $-D = \ln(X/s_{t_0}) - r(T - t_0)$. The right side of (6) is the minimum Bayes risk R_B for the hypotheses F_0^* and F_1^* under 0-1 loss with probabilities, respectively, π_0 and π_1 . The value for which the posterior densities of F_0^* and F_1^* are equal determines R_B .

b) From a) it follows that

$$C = (1 - R_B)s_{t_0} - R_B X e^{-r(T-t_0)} = s_{t_0} - R_B(s_{t_0} + X e^{-r(T-t_0)}) \quad (7)$$

$$= s_{t_0}[1 - F_1^*(-D)] - X e^{-r(T-t_0)}[1 - F_0^*(-D)]. \quad (8)$$

Thus, C can be called Bayes (B-) price, denoted also by $\tilde{C}_{B,t_0}(P^*)$.

We revisit Schachermayer’s (2008) “toy” example for an illustration.

Example 3.1 In Schachermayer's (2008) "toy" example, at time $t_0 = 0$ the stock has price $s_0 = 1$ USD and under martingale probability P^* ,

$$S_1 = \begin{cases} 2 \text{ USD} & \text{with prob. } P^*(S_1 = 2) = 1/3, \\ .5 \text{ USD} & \text{with prob. } P^*(S_1 = .5) = 2/3, \end{cases}$$

and $E_{P^*}S_1 = 1$ USD.

For strike price $X = 1$ USD, the price of the European option with maturity $T = 1$ and fixed interest $i = 0$ is

$$E_{P^*}(S_1 - 1)_+ = 1/3 \text{ USD}.$$

Let d be a generic exercise barrier like that used for Bayes risk in (6). To obtain the B-price for P^* maximize over d

$$s_0 E_{P^*} \frac{S_1}{E_{P^*} S_1} I(S_1 \geq d) - X P^*(S_1 \geq d) \quad (9)$$

$$= E_{P^*} S_1 I(S_1 \geq d) - P^*(S_1 \geq d) = \begin{cases} 1 - 1 = 0 \text{ USD}, & \text{for } 0 \leq d \leq .5 \\ 2 \cdot \frac{1}{3} - 1 \cdot \frac{1}{3} = \frac{1}{3} \text{ USD} & \text{for } .5 < d \leq 2. \end{cases}$$

Thus, B-price is the "fair" price $\frac{1}{3}$ USD.

Corollary 3.1 In addition to the assumptions used in Proposition 3.1, assume that F_0^* and F_1^* are location-scale cumulative distribution functions, i.e.

$$F_0^*(y) = G_0\left(\frac{y - \theta_0}{\sigma_0}\right), \quad F_1^*(y) = G_1\left(\frac{y - \theta_1}{\sigma_1}\right), \quad (10)$$

$\theta_i \in R$, $\sigma_i > 0$, $i = 0, 1$, and that $G_i(x) = 1 - G_i(-x)$, $x \in R$, $i = 0, 1$. Then, the Bayes price of the call option is

$$C = \tilde{C}_{B,t_0}(P^*) = s_{t_0} G_1\left(\frac{D + \theta_1}{\sigma_1}\right) - X e^{-r(T-t_0)} G_0\left(\frac{D + \theta_0}{\sigma_0}\right); \quad (11)$$

where

$$D = \ln(s_{t_0}/X) + r(T - t_0). \quad (12)$$

Remark 3.1 Under B-S-M assumptions, (11) is the B-S-M price; see (32).

Corollary 3.2 Under the assumptions in Proposition 3.1, let G be the game that results in loss $-(s_{t_0} - \tilde{C})$ with probability $1 - R_B$ and profit $\tilde{C} + X e^{-r(T-t_0)}$ with probability R_B (the Bayes risk). The value $\tilde{C} = \tilde{C}_{B,t_0}(P^*)$ makes G “fair”.

Remark 3.2 The results in Proposition 3.1 suggest for the American call option with strike X that can be exercised in $(t_0, T]$ the price $\tilde{C}_{B,A} = s_{t_0} - \inf_{t \in (t_0, T]} \{(s_{t_0} + X e^{-r(t-t_0)}) R_{B,t}\}$; $R_{B,t}$ is the Bayes risk under P^* of the European option with maturity t , $t_0 < t \leq T$. The obtained $\tilde{C}_{B,A}$ is a different form of the fair price of American option in Karatzas (1988, p. 50). By construction, $\tilde{C}_{B,A}$ does not allow arbitrage.

Conditions are given below for prices which follow Geometric model that guarantee the process $\{S_t/E_P S_t, t > 0\}$ is a martingale under P . In the sequel it is seen that these conditions hold when the stock price process is modeled by a Geometric Brownian motion or Hyperbolic Levy motion.

Lemma 3.1 *For the stock price process $\{S_t, t > 0\}$ on the probability space (Ω, \mathcal{F}, P) assume that*

$$S_t = s_0 e^{\mu t + V_t}, \quad t \geq 0, \quad (13)$$

with $\mu \in R$, $V_0 = 0$ and $\{V_t, t > 0\}$ having stationary and independent increments. Then, $E_P e^{V_t} = M^t$, $M > 0$, and the mean-adjusted prices $\{S_t/E_P S_t, t > 0\}$ are martingale under P .

4 $\tilde{C}_{B,t_0}(P)$ for Geometric Brownian motion

It is seen that $\tilde{C}_{B,t_0}(P)$ is the B - S - M -price C ; there is no need to determine P^* . Additional justification is initially provided for the use of discounting factor $A^{-1}(t_0, T) = s_{t_0}/E_P S_T$. Recall that in the B - S - M assumptions, the stock price process $\{S_t, t > 0\}$ on (Ω, \mathcal{F}, P) satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t > 0, \quad (14)$$

with $\{W_t, t > 0\}$ one-dimensional standard Brownian motion and $\{\mathcal{F}_t, t > 0\}$ the natural filtration.

In Musiela and Rutkowski (1997, p. 110-111) it is shown that for $t < T$,

$$E_P(S_T | \mathcal{F}_t) = E_P(S_T | S_t) = S_t e^{\mu(T-t)}. \quad (15)$$

The coefficient $e^{\mu(T-t)}$ describes the evolution of the price process from t to T and by taking expected values in (15) it follows that the discounting factor

from T to t is

$$e^{-\mu(T-t)} = \frac{E_P S_t}{E_P S_T}. \quad (16)$$

When the starting time is t_0 , equation (14) has analytic solution

$$S_t = s_{t_0} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma W_{t-t_0}\right\}, \quad t > t_0; \quad (17)$$

s_{t_0} is the share's price at t_0 .

From (16) and (17), to discount the share's price from T at t_0 we use

$$A^{-1}(t_0, T) = \frac{E_P S_{t_0}}{E_P S_T} = \frac{s_{t_0}}{E_P S_T}.$$

Definition 4.1 *For densities f and g on the real line, their Hellinger distance $H(f, g)$ is defined by*

$$H^2(f, g) = \int_R \{\sqrt{f(x)} - \sqrt{g(x)}\}^2 dx. \quad (18)$$

We then have the following proposition.

Proposition 4.1 *Under (A) and the B-S-M assumptions, discounting S_T with $A^{-1}(t_0, T)$ and cash with $e^{-r(T-t_0)}$ and defining F_0, f_0, f_1, F_1 for P as in (4) and (5), it is shown that the Bayes price $\tilde{C}_{B,t_0}(P)$ of the European call option is the B-S-M price. The Hellinger distance $H(f_1, f_0)$ increases with the volatility.*

Remark 4.1 *When the price process is a Geometric Brownian motion under P , from (17) for $t > u > t_0$ it holds*

$$E_P[S_t/E_P S_t | \mathcal{F}_u] = e^{-.5\sigma^2(u-t_0)+\sigma W_u} E_P[e^{-.5\sigma^2(t-u)+\sigma W_{t-u}}] = S_u/E_P S_u,$$

i.e. the mean-adjusted prices $\{S_t/E_P S_t, t > t_0\}$ are a martingale under P . Alternatively, observe that $E_P e^{\sigma W_t} = (e^{.5\sigma^2})^t$ and Lemma 3.1 holds with $M = e^{.5\sigma^2}$.

5 B -prices for other models

It is widely known that the distribution of the logarithm of price returns deviates from normality and the constant volatility assumption is often violated. Thus, researchers use also normal mixtures, distributions with heavier than normal tails, hyperbolic returns etc. B -price of the call option can be informative in these situations.

Assume that $t_0 = 0$ and that the stock price follows the model

$$S_t = s_0 e^{\mu t + X_t}, EX_t = 0, t > 0. \quad (19)$$

It is seen below that when X_t in (19) is a normal mixture, mean-adjusted prices are not martingale under P . However, for small t -values they “nearly” are and in a 2-normal mixture example it is observed that B - S - M price, obtained assuming X_t is Brownian motion with the same mean and variance, is often larger than B -price and the mixture of B - S - M prices obtained for each normal in the mixture. When X_t is a hyperbolic Lévy motion the martingale probability is not unique but mean-adjusted prices are martingale under P

and a “fair” B -price is obtained.

5.1 The normal mixture model

When X_t in (19) follows a normal mixture,

$$\mathcal{L}(X_t|P) = \sum_{i=1}^m p_i \mathcal{N}(0, a_i^2 t), \quad (20)$$

$0 < p_i < 1$, $a_i > 0$, $i = 1, \dots, m$, $\sum_{i=1}^m p_i = 1$. Then,

$$E_P S_t = s_0 e^{\mu t} E_P e^{X_t} = s_0 e^{\mu t} \sum_{i=1}^m p_i e^{\frac{a_i^2 t}{2}},$$

$$\frac{S_t}{E_P S_t} = e^{X_t - \ln \sum_{i=1}^m p_i e^{\frac{a_i^2 t}{2}}}, \text{ and}$$

$$f_0 = \mathcal{L}(\ln \frac{S_T}{E_P S_T} | P) = \sum_{i=1}^m p_i \mathcal{N}(-\ln G_T, a_i^2 T)$$

with

$$G_T = \sum_{i=1}^m p_i e^{\frac{a_i^2 T}{2}}, \quad t_0 = 0. \quad (21)$$

Recall that $f_1(y) = e^y f_0(y)$ and observe that

$$E_{f_0} e^Y = \sum_{i=1}^m p_i e^{-\ln G_T + \frac{a_i^2 T}{2}} = 1.$$

Then,

$$\begin{aligned} f_1(y) &= e^y f_0(y) = \sum_{i=1}^m p_i \frac{1}{a_i \sqrt{2\pi T}} e^{-\frac{y^2 + (\ln G_T)^2 + 2y(\ln G_T - a_i^2 T)}{2a_i^2 T}} \\ &= \sum_{i=1}^m p_i e^{\frac{-2a_i^2 T \ln G_T + a_i^4 T^2}{2a_i^2 T}} \mathcal{N}(-\ln G_T + a_i^2 T, a_i^2 T) = \sum_{i=1}^m \frac{p_i e^{\frac{a_i^2 T}{2}}}{G_T} \mathcal{N}(-\ln G_T + a_i^2 T, a_i^2 T). \end{aligned}$$

From (8), B -price of the European call option is

$$\tilde{C}_{B,0}(P) = s_0 \sum_{i=1}^m q_i \Phi\left(\frac{D - \ln G_T + a_i^2 T}{a_i \sqrt{T}}\right) - X e^{-rT} \sum_{i=1}^m p_i \Phi\left(\frac{D - \ln G_T}{a_i \sqrt{T}}\right), \quad (22)$$

with $D = \ln(s_0/X) + rT$, $q_i = \frac{p_i e^{\frac{a_i^2 T}{2}}}{G_T}$, $i = 1, \dots, m$ and G_T as in (21). We observe that $\{S_t/E_P S_t, t > 0\}$ are not martingale for $m > 1$. However, for small t ($> u$) they “nearly” are since

$$E_P(S_t/E_P S_t | \mathcal{F}_u) = \frac{S_u}{E_P S_u} \frac{\sum_{i=1}^m p_i e^{a_i^2 u} \sum_{i=1}^m p_i e^{a_i^2 (t-u)}}{\sum_{i=1}^m p_i e^{a_i^2 t}} \approx_{\text{small } t} \frac{S_u}{E_P S_u}.$$

In Example 5.1, B -price is computed for small T and is compared with the B - S - M price for Geometric Brownian motion with the same variance and the corresponding mixture of B - S - M prices obtained under the mixture model.

Example 5.1 Assume that prices $\{S_t, t > 0\}$ follow model (19) and $m = 2$ in the mixture model (20), $p_1 = p = 1 - p_2$. We examine the effect of contaminations of the Geometric Brownian motion in the B - S - M price by comparing it with the B -price obtained under the mixture distribution for small T -values. We use $i = .04, .08$, $t_0 = 0$ and small but also larger values $T = .03, .05, .1, .15, .2, .5$ (in years), $s_0 = 60$ \$, $X = 70$ \$, $a_1 = 1$, $a_2 = 1.05, 1.2, 2, 4$ and the mixing coefficient p takes values $j/50, j = 1, \dots, 50$. The B - S - M price is obtained assuming prices follow Geometric Brownian motion with variance $pa_1^2 + (1-p)a_2^2$. The results in Tables 1 and 2 indicate that the B -price is often smaller than the B - S - M price when T is small; this does not hold

when $T = .5$. It has been observed, for example, that B - S - M price is larger than B -price and the same mixture of B - S - M prices obtained for each model in the mixture; $t = 0$, $T = .02$, $s = 60$, $X = 50$, $i = .08$, $K = 50$, $a_1 = 1$, $a_2 = 3$.

HOW OFTEN B-PRICE $<$ B-S-M PRICE, $i=.04$				
T (in years)	a2=1.05	a2=1.2	a2=2	a2=4
.03	0.01960784	0.01960784	0.9607843	0.4509804
.05	0.9607843	0.9803922	0.9607843	0.2352941
.1	0.9607843	0.9803922	0.7647059	0.01960784
.15	0.9803922	0.9803922	0.4313725	0.01960784
.2	0.9607843	0.9607843	0.1568627	0
.5	0.01960784	0.01960784	0.01960784	0.01960784

TABLE 1

HOW OFTEN B-PRICE < B-S-M PRICE, $i=.08$				
T (in years)	a2=1.05	a2=1.2	a2=2	a2=4
.03	0.01960784	0.1764706	1	0.4705882
.05	0.9803922	0.9803922	1	0.2745098
.1	0.9607843	0.9607843	0.8039216	0.01960784
.15	0.9607843	0.9803922	0.4313725	0
.2	0.9803922	0.9607843	0.1960784	0
.5	0.01960784	0.03921569	0.01960784	0.01960784

TABLE 2

5.2 The Hyperbolic Lévy motion model

Barndorff-Nielsen (1977) introduced the family of hyperbolic continuous distributions with logarithmic densities hyperbolas. Barndorff-Nielsen and Halgreen (1977) showed these are infinitely divisible. From empirical findings on stock returns, Eberlein and Keller (1995) (E & K) considered the Lévy process $\{Z_t, t > 0\}$, defined by the infinitely divisible hyperbolic distribution that is symmetric and centered with density

$$h(x; \zeta, \delta) = \frac{1}{2\delta K_1(\zeta)} e^{-\zeta \sqrt{1 + (\frac{x}{\delta})^2}}, \quad x \in R; \quad (23)$$

K_1 is the modified Bessel function of the third kind with index 1. The process $\{Z_t, t > 0\}$ has stationary, independent increments such that $Z_0 = 0$, Z_1 has density $h(x; \zeta, \delta)$ and characteristic function $\phi(u; \zeta, \delta)$, and Z_t has density

$$f_t(x; \zeta, \delta) = \frac{1}{\pi} \int_0^\infty \cos(ux) \phi^t(u; \zeta, \delta) du.$$

E & K called $\{Z_t, t > 0\}$ hyperbolic Lévy motion and used (19) with $\mu_t = 0$ to model stock prices,

$$S_t = s_0 e^{Z_t}, \quad t > 0. \quad (24)$$

E & K noticed that for model (24) there is no unique martingale probability and obtained a price for the European call option under a martingale probability using the Esscher transform of the process. Recent Fourier transform valuation formulas for Lévy and other models and securities can be found in Eberlein, Glau and Papantoleon (2010).

A “fair” B -price is now obtained under P complementing prices obtained using the Esscher transform (Gerber and Shiu, 1994).

Proposition 5.1 *a) The mean-adjusted prices $\{S_t/ES_t, t > 0\}$ are martingale under P .*

b) B -price is

$$\tilde{C}_{B,0}(P) = s_0 \int_{\ln(X/s_0)-rT}^\infty e^x f_T(x+T \ln M) dx - X e^{-rT} \int_{\ln(X/s_0)-rT}^\infty f_T(x+T \ln M) dx.$$

6 Concluding Remarks

A purely statistical interpretation of the price C of the European call option has been provided from new formula (1). Advantages of B -prices include:

a) When mean-adjusted stock prices $\{S_t/E_P S_t, t_0 \leq t \leq T\}$ are martingale under P , C can be obtained without prior determination of P^* as in sections 4 and 5.2.

b) For small T -values, $\{S_t/E_P S_t\}$ is often nearly a martingale under P and an approximation for C can be obtained as in section 5.1.

c) In (1), one could model R_B , estimate the unknown parameters using C market-values and use the so-obtained estimate to derive other C -values.

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Appendix

Proof of Proposition 3.1 *a)* We start by proving the last equality in (6).

Let

$$R(d) = \pi_1 F_1^*(\ln \frac{d}{ES_T}) + \pi_0 [1 - F_0^*(\ln \frac{d}{ES_T})]. \quad (25)$$

In the right side of (25), regions $(-\infty, \ln \frac{d}{ES_T}]$ and $(\ln \frac{d}{ES_T}, +\infty)$ are a partition of the real line so they determine a decision function and $R(d)$ is Bayes risk for the estimation problem of F_1^* and F_0^* with 0–1 loss and prior probabilities π_1 and π_0 respectively. To minimize R_d consider its first derivative,

$$\begin{aligned} R'(d) &= \pi_1 F_1^{*'}(\ln \frac{d}{ES_T})g(d) - \pi_0 F_0^{*'}(\ln \frac{d}{ES_T})g(d) \\ &= g(d)\pi_0 f_0^*(\ln \frac{d}{ES_T}) \left(\frac{\pi_1 f_1^*(\ln \frac{d}{ES_T})}{\pi_0 f_0^*(\ln \frac{d}{ES_T})} - 1 \right) \\ &= g(d)\pi_0 f_0^*(\ln \frac{d}{ES_T}) \left(\frac{\pi_1}{\pi_0} \frac{d}{ES_T} - 1 \right), \end{aligned} \quad (26)$$

where (26) follows from (5); $g(d)$ is a term due to the first derivative.

Thus, from (5) and (26)

$$\begin{aligned} R'(d_B) = 0 &\iff \pi_1 f_1^*(\ln \frac{d_B}{ES_T}) = \pi_0 f_0^*(\ln \frac{d_B}{ES_T}) \\ &\iff d_B = \frac{\pi_0 ES_T}{\pi_1} = X \frac{e^{-r(T-t_0)}}{s_{t_0}/ES_T}. \end{aligned} \quad (27)$$

It also holds

$$R''(d_B) = g(d_B)\pi_1 f_0^*(\ln \frac{d_B}{ES_T}) \frac{1}{ES_T} > 0$$

and the minimum Bayes risk

$$R_B = R(d_B) = \pi_1 F_1^* \left(\ln \frac{X e^{-r(T-t_0)}}{s_{t_0}} \right) + \pi_0 [1 - F_0^* \left(\ln \frac{X e^{-r(T-t_0)}}{s_{t_0}} \right)]. \quad (28)$$

To prove the first equality in (6), note that since interest discounted stock prices are martingale under P^* ,

$$e^{-r(T-t_0)} E S_T = s_{t_0} \rightarrow E S_T = s_{t_0} e^{r(T-t_0)}. \quad (29)$$

Use (29) to express the “fair” price C of the option (in (3)) using F_1^* and F_0^* ,

$$\begin{aligned} C &= e^{-r(T-t_0)} E S_T I(S_T > X) - X e^{-r(T-t_0)} P^*(S_T > X) \\ &= s_{t_0} E e^{\ln \frac{S_T}{E S_T}} I\left(\ln \frac{S_T}{E S_T} > \ln \frac{X}{s_{t_0} e^{r(T-t_0)}}\right) - X e^{-r(T-t_0)} P^*\left(\ln \frac{S_T}{E S_T} > \ln \frac{X}{s_{t_0} e^{r(T-t_0)}}\right) \\ &= s_{t_0} [1 - F_1^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right)] - X e^{-r(T-t_0)} [1 - F_0^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right)], \end{aligned} \quad (30)$$

and then

$$s_{t_0} - C = s_{t_0} F_1^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right) + X e^{-r(T-t_0)} [1 - F_0^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right)] \leq s_{t_0} + X e^{-r(T-t_0)},$$

or

$$\frac{s_{t_0} - C}{s_{t_0} + X e^{-r(T-t_0)}} = \pi_1 F_1^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right) + \pi_0 [1 - F_0^* \left(\ln \frac{X}{s_{t_0}} - r(T-t_0) \right)]. \quad (31)$$

The result follows from (28) and (31).

b) Follows from part a) and (28). □

Proof of Corollary 3.1 Follows from (8) and the assumption $G_i(x) = 1 - G_i(-x)$, $x \in R$, $i = 0, 1$. \square

Proof of Corollary 3.2 Follows from Proposition 3.1 rearranging (7) to obtain

$$R_B(C + Xe^{-r(T-t_0)}) - (s_{t_0} - C)(1 - R_B) = 0. \quad \square$$

Proof of Lemma 3.1 From model (13) it holds

$$E_P e^{V_t} = (E_P e^{V_1})^t = M^t \text{ and}$$

$$E_P S_t = s_0 e^{\mu t} E_P e^{V_t} = s_0 e^{\mu t} M^t,$$

and therefore, for $t > u$, from stationarity and independence of $\{V_t\}$ increments

$$E_P\left(\frac{S_t}{E_P S_t} \middle| \mathcal{F}_u\right) = \frac{e^{V_u}}{M^t} E_P e^{V_{t-u}} = \frac{e^{V_u}}{M^u} = \frac{S_u}{E_P S_u}. \quad \square$$

Proof of Proposition 4.1 From Remark 3.1, the value of the writer's expected cost at T discounted at t_0 is given by (30) with P^* replaced by P and therefore (8) holds with F_0 and F_1 instead of F_0^* and F_1^* . From (17) it follows that $E_P S_T = s_{t_0} \exp\{\mu(T - t_0)\}$, and

$$\mathcal{L}(\ln \frac{S_T}{E_P S_T} | P) = \mathcal{N}(-\frac{\sigma^2}{2}(T - t_0), \sigma^2(T - t_0)) = F_0;$$

$\mathcal{N}(\theta, \tau^2)$ is used to denote a normal distribution with mean θ and variance τ^2 .

From (5) it follows that

$$f_1(x) = e^x \frac{1}{\sigma \sqrt{T - t_0}} \phi\left(\frac{x + .5\sigma^2(T - t_0)}{\sigma^2(T - t_0)}\right) = \frac{1}{\sigma \sqrt{T - t_0}} \phi\left(\frac{x - .5\sigma^2(T - t_0)}{\sigma^2(T - t_0)}\right)$$

i.e. $F_1 = \mathcal{N}(\frac{\sigma^2(T-t_0)}{2}, \sigma^2(T-t_0))$; ϕ denotes standard normal density.

From (11), with $G_1 = G_0 = \mathcal{N}(0, 1)$, $\theta_1 = \frac{\sigma^2(T-t_0)}{2}$, $\theta_0 = -\frac{\sigma^2(T-t_0)}{2}$, it follows that

$$\tilde{C}_{B,t_0}(P) = s_{t_0}\Phi(d_1) - Xe^{-r(T-t_0)}\Phi(d_2), \quad (32)$$

$$d_j = \frac{\ln(s_{t_0}/X) + r(T-t_0) + (-1)^{1+j} \frac{\sigma^2}{2}(T-t_0)}{\sigma\sqrt{T-t_0}} = \frac{D + (-1)^{1+j} \frac{\sigma^2}{2}(T-t_0)}{\sigma\sqrt{T-t_0}}, \quad j = 1, 2;$$

Φ is the cumulative distribution of a standard normal, $\sigma > 0$, D is determined

in (12). It follows that

$$H^2(f_1, f_0) = 2(1 - e^{-\frac{\sigma^2(T-t_0)}{8}}). \quad \square$$

Proof of Proposition 5.1 *a)* By stationarity and independence of the increments, the moment generating function of Z_t (E& K, p. 297, equation 30)

$$E_P e^{Z_t} = (E_P e^{Z_1})^t = \left(\frac{\zeta}{K_1(\zeta)} \frac{K_1(\sqrt{\zeta^2 - \delta^2})}{\sqrt{\zeta^2 - \delta^2}} \right)^t := M^t, \quad \delta < \zeta,$$

$$E_P S_t = s_0 E_P e^{Z_t} = s_0 M^t.$$

The result follows from Lemma 3.1.

b) Compute B -price as described in section 2. We then have

$$\ln \frac{S_T}{E_P Z_T} = Z_T - T \ln M \sim_P f_T(x + T \ln M) = f_0(x)$$

and since $t_0 = 0$ from (8) the B -price is

$$\tilde{C}_{B,0}(P) = s_0 \int_{\ln(X/s_0) - rT}^{\infty} e^x f_T(x + T \ln M) dx - X e^{-rT} \int_{\ln(X/s_0) - rT}^{\infty} f_T(x + T \ln M) dx.$$

□

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